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A simple explanation of the classic hydrostatic paradox

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Abstract

An interesting problem in fluid mechanics, with significant educational importance, is the classic hydrostatic paradox. The hydrostatic paradox states the fact that in different shaped containers, with the same base area, which are filled with a liquid of the same height, the applied force by the liquid on the base of each container is exactly the same. However, if the shape of the container is different, the amount of the liquid (and as a consequence the weight) can greatly vary. In this paper, a simple explanation of the hydrostatic paradox, specifically designed and implemented for educational purposes regarding secondary education, is provided.

Introduction

The paradoxes in physics have been reported in several situations [1–4]. Generally, the physical paradoxes represent peculiar physical situations that can illustrate how some principles act in some contexts [5]. A very significant area of physics which provides many paradoxical situations is fluid mechanics [5–7]. In this paper, a specific situation in which a liquid can apply force greater or smaller than its weight on the bottom of the container in which it remains at rest is presented and analysed in detail, concerning physics lessons in secondary education. The so called classic hydrostatic paradox was firstly presented by Blaise Pascal in 1646. In particular, Pascal inserted a long tube, vertical to a barrel which was filled with water. The height of the tube was approximately 10 m. After the insertion of the tube into the barrel, the tube was also filled with water. Despite the fact that the weight of the water did not increase significantly (the tube was very thin comparing to the barrel), the applied force on the barrel's sides led to its destruction. The explanation of the above experiment was based on the fact that the hydrostatic pressure is increased proportionally to the depth below the free surface of the water, regardless of the shape of the container in which the liquid rests. The simple equation which relates the hydrostatic pressure and the depth below the liquid's free surface (Stevin's law) is [5, 8, 9]:

$$P = \rho \cdot g \cdot h \tag{1}$$

Where, P is the hydrostatic pressure of the liquid (which is at rest) in a certain point of depth h, ρ is the liquid's density and g the acceleration of gravity. Hence, the force on each elementary area dA of the barrel increases proportionally to the depth below the liquid's free surface. However, according to the above statement, the force applied on the bottom of a container can be significantly greater than the liquid's weight depending on the shape of the container. This paradox is generally solved by considering the forces on the other sides of the container [5]. The hydrostatic force is applied perpendicular to all the walls of the container in which the liquid remains at rest. Hence, the lateral forces produce a contribution which must be summed or subtracted to the contribution at the bottom on the vessel. Thus, the correct sum or difference, of all contributions provides a force

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always equal to the weight of the liquid [5]. In this paper, a complete presentation and explanation of the hydrostatic paradox was developed aiming at undergraduate students regarding physics class.

The classic experiment of hydrostatics

Initially, assume a barrel in which a thin tube is inserted vertically on its top side (figure 1(a)). The system is filled with liquid. The liquid's weight can be easily calculated by the simple equation:

$$w = m \cdot g = \rho \cdot V \cdot g \tag{2}$$

In equation (2), m is the liquid's mass, g is the acceleration due to gravity and ρ, V are the liquid's density and volume, respectively. The liquid's volume is equal to the volume of the barrel (V_B) and the tube (V_T) as it is presented in figure 1(a):

$$V = V_\mathrm{B} + V_\mathrm{T} = A_\mathrm{B} \cdot h_\mathrm{B} + A_\mathrm{T} \cdot h_\mathrm{T} \tag{3}$$

Where, $A_\mathrm{B}, A_\mathrm{T}, h_\mathrm{B}, h_\mathrm{T}$ are the cross-sectional areas and the heights of the barrel and the tube, respectively. By substituting equation (3) in equation (2), the weight of the liquid can be calculated in terms of the geometrical characteristics of the container (barrel and tube):

$$w = m \cdot g = \rho \cdot g \cdot (A_\mathrm{B} \cdot h_\mathrm{B} + A_\mathrm{T} \cdot h_\mathrm{T}) \Rightarrow w = \rho \cdot g \cdot h_\mathrm{B} \cdot A_\mathrm{B} + \rho \cdot g \cdot h_\mathrm{T} \cdot A_\mathrm{T} \tag{4}$$

Equation (4) provides the liquid's weight as a function of the barrel's and tube's height and cross-sectional areas. The next step is the calculation of the applied force on the barrel's bottom:

$$F_\mathrm{B} = P \cdot A_\mathrm{B} = \rho \cdot g \cdot (h_\mathrm{B} + h_\mathrm{T}) \cdot A_\mathrm{B} \Rightarrow F_\mathrm{B} = \rho \cdot g \cdot h_\mathrm{B} \cdot A_\mathrm{B} + \rho \cdot g \cdot h_\mathrm{T} \cdot A_\mathrm{B} \tag{5}$$

The comparison of the equations (4) and (5) leads to the result that the total force on the bottom of the container is greater than the liquid's weight ($F_\mathrm{B} > w$) due to the fact that $A_\mathrm{B} > A_\mathrm{T}$. For example, assume that $\rho = 1000\,\mathrm{kg\,m^{-3}}$, $g = 9.8\,\mathrm{m\,s^{-2}}$, $h_\mathrm{T} = 9.5\,\mathrm{m}$, $h_\mathrm{B} = 0.5\,\mathrm{m}$, $A_\mathrm{B} = 1\,\mathrm{m}^2$ and $A_\mathrm{T} = 10^{-3}\,\mathrm{m}^2$. In this case, the liquid's weight is $w = 4993.1\,\mathrm{N}$ and the total force on the barrel's bottom is $F_\mathrm{B} = 98.000\,\mathrm{N}$. Hence, the total force on the barrel's bottom is significantly bigger than its weight.

In addition to the above, assume a small area ($A = 10\,\mathrm{cm}^2 = 10^{-3}\,\mathrm{m}^2$) at $y = 0.25\,\mathrm{m}$ above the ground, then the force prior the insertion of the tube can be calculated,

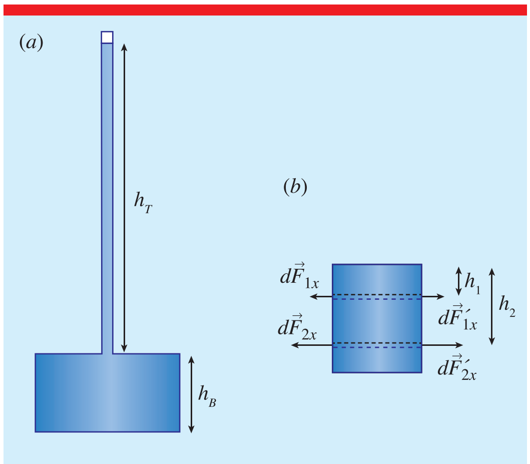

Figure 1. (a) A barrel in which a thin tube is inserted vertically on its top side. The system is filled with liquid (e.g. water). (b) If the column of the liquid is divided in small horizontal areas of elementary heights dy, the horizontal force for each horizontal area is zero. Hence, the total horizontal force on the tube–barrel system is zero.

$$F' = P \cdot A' = \rho \cdot g \cdot (h_\mathrm{B} - y) \cdot A' = 2.45\,\mathrm{N},$$

where $h_\mathrm{B} - y$ is the distance of the small area from the top of the barrel. Moreover, the total force post the insertion of the tube can be calculated,

$$F'' = P \cdot A' = \rho \cdot g \cdot (h_\mathrm{T} + h_\mathrm{B} - y) \cdot A' = 95.55\,\mathrm{N}$$

which is 39 times bigger that F'. The above calculation provides the cause of the destruction of the barrel after the insertion of the tube.

In order to explain the above paradox, the total force on the sides of the barrel–tube system will be calculated. The first step is the calculation of the total horizontal force on the sides of the system. If the liquid in the container is divided in small horizontal areas of elementary heights dy, the total horizontal force for each area is zero. For example in figure 1(b):

$$\mathrm{d}F_{1x}' - \mathrm{d}F_{1x} = \rho \cdot g \cdot h_1 \cdot \mathrm{d}A - \rho \cdot g \cdot h_1 \cdot \mathrm{d}A = 0$$

$$\mathrm{d}F_{2x}' - \mathrm{d}F_{2x} = \rho \cdot g \cdot h_2 \cdot \mathrm{d}A - \rho \cdot g \cdot h_2 \cdot \mathrm{d}A = 0$$

Where, dA represents an elementary area in which the force dF acts. Hence, the total horizontal force is zero, $\Sigma F_x = 0$.

The next step is the calculation of the total vertical force on the bottom and the upper side

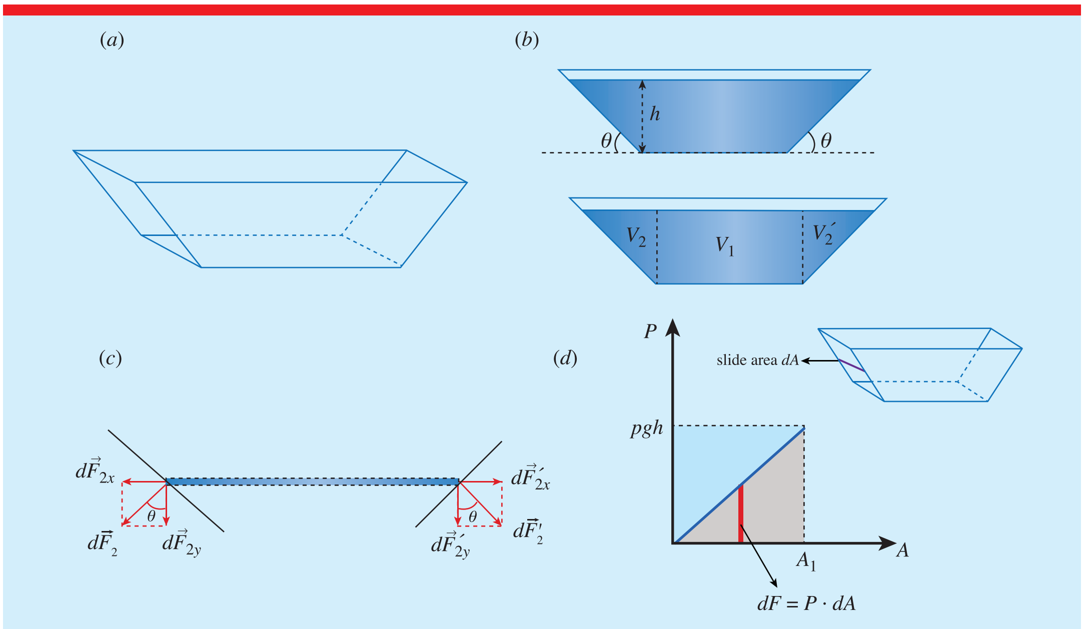

Figure 2. (a) Shape of the container, (b) the container is filled with liquid (e.g. water) and its volume can be calculated by summing the volumes V_1, V_2 and V_2' (c) if the column of the liquid is divided in small horizontal areas of elementary heights dy, the total horizontal force for each area is zero, (d) the total force on each side area of the container can be calculated from the area under the graph.

of the barrel. The total force applied on the upper side of the barrel is:

$$F'_{\mathrm{B}} = P' \cdot (A_{\mathrm{B}} - A_{\mathrm{T}}) = \rho \cdot g \cdot h_{\mathrm{T}} \cdot A_{\mathrm{B}} - \rho \cdot g \cdot h_{\mathrm{T}} \cdot A_{\mathrm{T}} \tag{6}$$

Hence, the total vertical force on the barrel which is equal to the total force (since the total horizontal force is zero) using also the result from equation (5) is:

$$\begin{aligned} \Sigma F_y = F_{\mathrm{B}} - F'_{\mathrm{B}} &= \rho \cdot g \cdot h_{\mathrm{B}} \cdot A_{\mathrm{B}} + \rho \cdot g \cdot h_{\mathrm{T}} \cdot A_{\mathrm{B}} \\ &- \rho \cdot g \cdot h_{\mathrm{T}} \cdot A_{\mathrm{B}} + \rho \cdot g \cdot h_{\mathrm{T}} \cdot A_{\mathrm{T}} \\ &\Rightarrow \Sigma F = \rho \cdot g \cdot h_{\mathrm{B}} \cdot A_{\mathrm{B}} + \rho \cdot g \cdot h_{\mathrm{T}} \cdot A_{\mathrm{T}} \end{aligned} \tag{7}$$

Thus, $\Sigma F = \Sigma F_y = w$.

As a conclusion, the hydrostatic paradox is clarified by calculating the total force on every side of the container in which the liquid remains at rest. In a container of any shape in general, the total force on all the container's sides and the bottom is always equal to the liquid's weight. The above statement is also proved in the next two more complex cases.

Two similar cases

Case 1: the liquid's weight exceeds the force on the bottom of the container

Assume a container with a shape as presented in figure 2(a), filled with liquid which remains at rest (figure 2(b)). As analysed in the previous section, the liquid's weight can be expressed in the form of equation (2). The liquid's volume can be calculated assuming a symmetric container as follows:

$$\begin{aligned} V &= A \cdot h + \frac{1}{2} \cdot A_1 \cdot h \cdot \cos\theta + \frac{1}{2} A_2 \cdot h \cdot \cos\theta \\ &= A \cdot h + A_1 \cdot h \cdot \cos\theta \end{aligned} \tag{8}$$

Where, A is the container's bottom area and A_1, A_2 are the side areas. Due to the symmetric container: $A_1 = A_2$. In addition, h is the height of the container and θ is the angle between each side and the linear ground.

The liquid's weight can be calculated by substituting equation (8) in equation (2):

$$\begin{aligned} w &= \rho \cdot g \cdot (A \cdot h + A_1 \cdot h \cdot \cos\theta) \Rightarrow w \\ &= \rho \cdot g \cdot h \cdot (A + A_1 \cdot \cos\theta) \end{aligned} \tag{9}$$

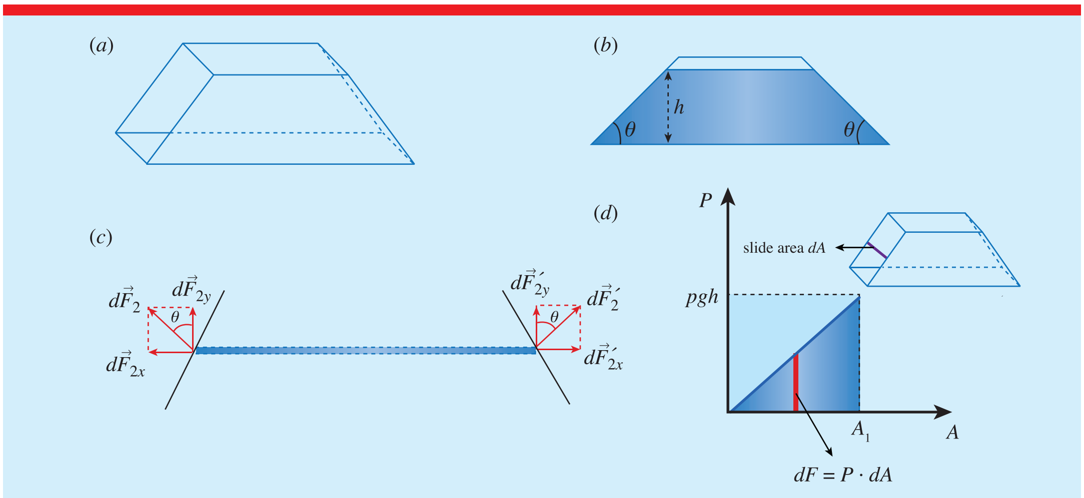

Figure 3. (a) Shape of the container, (b) the container is filled with liquid (e.g. water), (c) if the column of the liquid is divided in small horizontal areas of elementary heights dy, the total horizontal force for each area is zero, (d) the total force on each side area of the container can be calculated from the area under the graph.

Moreover, the total force applied on the container's bottom is:

$$F_1 = P \cdot A = \rho \cdot g \cdot h \cdot A < w$$

The next step is the calculation of the total force on each container's side. The total applied force perpendicular on each side can be calculated by the area under the graph *P*–*A* (see appendix) which is presented in figure 2(d):

$$F_2 = F_2' = \frac{\rho \cdot g \cdot h \cdot A_1}{2} \tag{10}$$

It must be clarified that no mention to the atmospheric force on the walls of the container (due to the atmospheric pressure) was examined due to the fact that it is applied equally to both sides of the walls.

In addition, the force on each side of the container can be analysed in two force components:

$$F_{2x} = F_{2x}' = \frac{\rho \cdot g \cdot h \cdot A_1}{2} \sin\theta$$

$$F_{2y} = F_{2y}' = \frac{\rho \cdot g \cdot h \cdot A_1}{2} \cos\theta$$

As mentioned above, if the column of the liquid is divided in small horizontal areas of elementary heights dy, the total horizontal force for each part is zero (figure 2(c)):

$$\begin{aligned} \mathrm{d}F_{2x}' - \mathrm{d}F_{2x} &= \rho \cdot g \cdot y \cdot \sin\theta \cdot \mathrm{d}A \\ &\quad - \rho \cdot g \cdot y \cdot \sin\theta \cdot \mathrm{d}A = 0 \\ &\Rightarrow F_{2x} - F_{2x}' = 0 \end{aligned}$$

The total vertical force on the container which is equal to the total force (since the total horizontal force is zero) can be calculated as follows:

$$\begin{aligned} &\Sigma F_y = F_1 + F_{2y} + F_{2y}' \\ &\quad \Rightarrow \Sigma F_y = \rho \cdot g \cdot h \cdot A + \frac{1}{2} \cdot \rho \cdot g \cdot A_1 \cdot h \cdot \cos\theta \\ &\quad + \frac{1}{2} \cdot \rho \cdot g \cdot A_2 \cdot h \cdot \cos\theta \\ &\quad \Rightarrow \Sigma F_y = \rho \cdot g \cdot h \cdot A + \rho \cdot A_1 \cdot g \cdot h \cdot \cos\theta \\ &\quad \Rightarrow \Sigma F_y = \rho \cdot g \cdot h \cdot (A + A_1 \cdot \cos\theta) \end{aligned} \tag{11}$$

Hence, the comparison of equations (9) and (11) proves that the total force on the sides and the bottom of the container is equal to the liquid's weight.

Case 2: the force on the bottom of the container exceeds liquid's weight

Assume a container with a shape as presented in figure 3(a), filled with liquid which remains at rest (figure 3(b)). The liquid's volume can be calculated assuming a symmetric container as follows:

$$V = A \cdot h - \frac{1}{2} \cdot A_1 \cdot h \cdot \cos\theta - \frac{1}{2} A_2 \cdot h \cdot \cos\theta$$
$$= A \cdot h - A_1 \cdot h \cdot \cos\theta \qquad (12)$$

Where, A is the container's bottom area, and A_1, A_2 are the side areas of the container. Due to the symmetric container: $A_1 = A_2$. In addition, h is the height of the container and θ is the angle between each side and the linear ground. Hence, by substituting equation (12) in equation (2):

$$w = \rho \cdot g \cdot (A \cdot h - A_1 \cdot h \cdot \cos\theta) \Rightarrow w$$
$$= \rho \cdot g \cdot h \cdot (A - A_1 \cdot \cos\theta) \qquad (13)$$

Moreover, the total force on the container's bottom is $F_1 = P \cdot A = \rho \cdot g \cdot h \cdot A > w$. The next step is the calculation of the total force on each container's side. The total force perpendicular on each side can be calculated by the area under the graph P–A which is presented in figure 3(d).

$$F_2 = F'_2 = \frac{\rho \cdot g \cdot h \cdot A_1}{2} \qquad (14)$$

However, the force on each side of the container can be analysed in two components:

$$F_{2x} = F'_{2x} = \frac{\rho \cdot g \cdot h \cdot A_1}{2} \sin\theta$$

$$F_{2y} = F'_{2x} = \frac{\rho \cdot g \cdot h \cdot A_1}{2} \cos\theta$$

As mentioned previously, if the column of the liquid is divided into small horizontal areas of elementary heights dy, the total horizontal force for each part is zero (figure 3(c)):

$$\mathrm{d}F'_2 - \mathrm{d}F_{2x} = \rho \cdot g \cdot y \cdot \sin\theta \cdot \mathrm{d}A$$
$$- \rho \cdot g \cdot y \cdot \sin\theta \cdot \mathrm{d}A = 0$$
$$\Rightarrow F_{2x} - F'_{2x} = 0$$

The total vertical force on the container which is equal to the total force (since the total horizontal force is zero) can be calculated as follows:

$$\Sigma F_y = F_1 - F_{2y} - F'_{2y}$$
$$\Rightarrow \Sigma F_y = \rho \cdot g \cdot h \cdot A - \frac{1}{2} \cdot \rho \cdot g \cdot A_1 \cdot h \cdot \cos\theta$$
$$- \frac{1}{2} \cdot \rho \cdot g \cdot A_2 \cdot h \cdot \cos\theta$$
$$\Rightarrow \Sigma F_y = \rho \cdot g \cdot h \cdot A$$
$$- \rho \cdot A_1 \cdot g \cdot h \cdot \cos\theta \Rightarrow \Sigma F_y$$
$$= \rho \cdot g \cdot h \cdot (A - A_1 \cdot \cos\theta) \qquad (15)$$

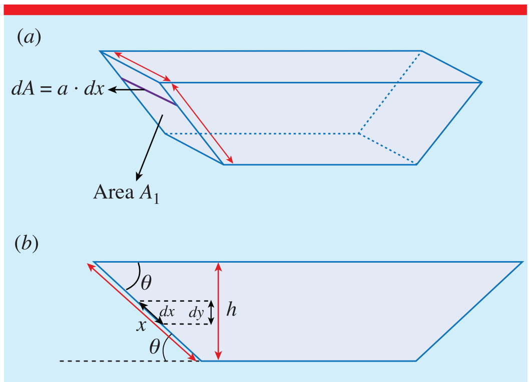

Figure A1. (a) Graphical representation of quantities: a, x, dA, A_1. (b) Graphical representation of quantities: θ, h, dx, dy.

Hence, the comparison of equations (13) and (15) proves that the total force on the sides and the bottom of the container is equal to the liquid's weight.

Conclusion

In this paper a total presentation and explanation of the classic hydrostatic paradox, regarding the needs of secondary education is provided. As it was thoroughly explained also from the two additional cases presented above, the paradox is generally solved by considering the sum of the forces on all sides of the container in which the liquid remains at rest.

Appendix

Due to the fact that this paper reflects the educational needs of secondary education, the extensive use of calculus was avoided. Thus, the total force perpendicular on each side of the containers in figures 2 and 3 was calculated using the graph $P - A$. The above method of calculation of physical quantities is familiar to the students of secondary education. Typical examples are the calculation of the area under the 'velocity–time' graph of a particle's motion across a straight line which is equal to the particle's displacement and the calculation of the area under the 'force–displacement' graph which provides the work done by the force applied on a moving body.

However, it must be noted that the total force perpendicular on each side of the containers in figures 2 and 3, can be also calculated using the integral [9–11]:

$$F = \int_A P \cdot \mathrm{d}A \tag{A.1}$$

According to figure A1(a), the elementary area dA is equal to: $\mathrm{d}A = a \cdot \mathrm{d}x$

However, according to figure A1(b), $x = \frac{h}{\sin\theta} \Rightarrow \mathrm{d}x = \frac{1}{\sin\theta}\mathrm{d}y$

In addition, the hydrostatic pressure at distance y below the liquid's free surface is:

$$P = \rho \cdot g \cdot y$$

Hence, equation (A.1) can be written in the form:

$$F = \int_0^h \rho \cdot g \cdot y \cdot \frac{a}{\sin\theta} \cdot \mathrm{d}y = \frac{\rho \cdot g \cdot a}{\sin\theta} \cdot \int_0^h y \cdot \mathrm{d}y$$
$$= \frac{\rho \cdot g \cdot a}{\sin\theta} \cdot \left[\frac{y^2}{2}\right]_0^h = \frac{\rho \cdot g \cdot a \cdot h^2}{2 \cdot \sin\theta}$$
$$= \rho \cdot g \cdot \frac{a \cdot h}{\sin\theta} \cdot \frac{h}{2} = \rho \cdot g \cdot a \cdot x \cdot \frac{h}{2} = \rho \cdot g \cdot A_1 \cdot \frac{h}{2}$$
$$= \frac{\rho \cdot g \cdot h \cdot A_1}{2}$$

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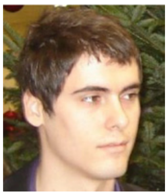

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